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Multiconnected neural network models

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Abstract. A generalisation of the Hopfield model which includes interactions between $p(>2)$ Ising spins is considered. The exact storage capacity behaves as $N^{p-1}/2(p-1)!\ln N$ when the number of nodes, N, is large. In the limit $p\to\infty$, the thermodynamics of the model can be solved exactly without using the replica method; at zero temperature, a solution which is completely correlated with the input pattern exists for $\alpha<\alpha_c$ where $\alpha_c\to\infty$ as $p\to\infty$ and this solution has lower energy than the spin-glass solution if $\alpha<\alpha_1=1/4\ln 2$ where the number of patterns $n=(2\alpha/p!)N^{p-1}$. For finite values of p, the correlation with the input pattern is not complete; for $p=3$ and 4, approximate values of α_c and α_1 are obtained and for $p\to\infty$ the replica symmetric approximation gives $\alpha_c\sim p/4\ln p$.

1. Introduction

Spin glasses provide models for storage and associative memory properties of neural networks. In particular, the Hopfield model (Hopfield 1982) describes the storage of n random patterns as N bit numbers. Although the number of patterns which can be stored perfectly behaves as $N/2\ln N$ (Weisbuch and Fogelman Soulie 1985, Bruce *et al* 1987) for large N, the model is expected to be useful as a pattern recognition model for finite values of the storage ratio $\alpha=n/N$; at low temperature (Amit *et al* 1985b, 1986), there exists a metastable state which is highly correlated with a particular input pattern provided $\alpha<\alpha_c$ and, therefore, a close representation of the input vector will be recovered (for $\alpha<\alpha_c$) provided one starts the iteration within the basin of attraction of the correlated state. For $\alpha<\alpha_1$, the correlated state becomes the ground state. The phase diagram, however, was calculated in the replica symmetric approximation and it is not known whether any new effects arise when replica symmetry breaking is included (although the effect on the values of α_1 and α_c is expected to be small (Crisanti *et al* 1986)).

A natural generalisation of the Hopfield model is to replace the two-spin interaction by a polynomial interaction of degree $p>2$ in the Ising spins (Peretto and Niez 1986). Introduction of many-spin interactions should also be useful in other pattern recognition problems in order to describe correlations of higher order than two. Although other methods of dealing with higher-order correlations exist, for example using two-spin interactions and hidden neurons (Ackley *et al* 1985, Rumelhart *et al* 1985), the learning algorithms for the interactions usually involve long timescales and therefore more direct methods are of interest.

In this paper, the following generalisation of the Hopfield model, where the Hamiltonian is a monomial of degree p in the Ising spins, will be considered:

$$\mathcal{H}=-\sum_{i_1<\ldots<i_p} J_{i_1,\ldots,i_p}S_{i_1}S_{i_2}\ldots S_{i_p} \tag{1}$$

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where the S_i, $i = 1, \ldots, N$, are the Ising spins and where the interactions are given by a generalised Hebb rule,

$$J_{i_1 \ldots i_p} = \frac{p!}{N^{p-1}} \sum_{\mu=1}^{n} \xi_{i_1}^{\mu} \xi_{i_2}^{\mu} \ldots \xi_{i_p}^{\mu} \tag{2}$$

where the $\{\xi_i^{\mu} = \pm 1\}$, $\mu = 1, \ldots, n$, are the random patterns one wants to store. The spins are updated according to the rule that S_i becomes $+1$ if the local magnetic field

$$H_i = \sum_{i_2 < i_3 < \ldots < i_p} J_{i, i_2, i_3, \ldots, i_p} S_{i_2} S_{i_3} \ldots S_{i_p} \tag{3}$$

is greater than zero, and becomes -1 otherwise. The dynamics therefore takes one, depending on the initial conditions, to a local minimum of the energy defined in equation (1) by $\mathcal{H}$.

The paper is organised as follows: in § 2, perfect storage properties of the network will be discussed. The number of patterns which can be stored without error increases with p and N as $N^{p-1}/2(p-1)! \ln N$ for large N. As in the two-spin case, the thermodynamics of the model predicts two critical values of the storage ratio $\alpha = p! n / 2N^{p-1}$; $\alpha_c(p, T)$ corresponding to the existence of a metastable state close to the input pattern and $\alpha_1(p, T)$ corresponding to this state having a lower energy than the spin-glass state (where T is the temperature). The storage capacity allowing for a small fraction of errors is therefore larger than the perfect storage capacity and increases with N as $2\alpha_c(p, 0) N^{p-1}/p!$ for large N. The transition at α_c is first order as in the two-spin case but the mechanism is different. The large-p limit of the model is exactly soluble and its phase diagram will be determined in § 3 (the limit $p \to \infty$ is taken after the limit $N \to \infty$). In particular, $\alpha_c(p, 0) \to \infty$ as $p \to \infty$ while $\alpha_1(p, 0) \to 1/4 \ln 2$ in this limit. In § 4, the replica formalism for the thermodynamics of the model will be introduced; replica symmetry breaking effects can be calculated exactly in the large-p limit and, for finite values of p, approximate values for α_c and α_1 will be obtained.

2. Perfect storage

If a particular input vector $\{\xi_i^1\}$ is to be stored without error then one requires that it is a stable state of the dynamics described by equation (3). This means that the local magnetic field H_i for $\{S_i\} = \{\xi_i^1\}$ must have the same sign as the spin ξ_i^1 at each site i. Therefore the quantities

$$R_i = \sum_{i_2 < i_3 < \ldots < i_p} J_{i, i_2, \ldots, i_p} \xi_i^1 \xi_{i_2}^1 \ldots \xi_{i_p}^1 \tag{4}$$

must be positive at each site i. The right-hand side of equation (4) can be divided into two parts: a signal term coming from the contribution of pattern 1 to $J_{i, i_2, \ldots, i_p}$ which is equal to p and therefore favours its recall and an interference term coming from the contribution to $J_{i, i_2, \ldots, i_p}$ of all other patterns which has mean zero and Gaussian fluctuations of variance $\sqrt{2\alpha p}$ where the number of patterns

$$n = 2\alpha N^{p-1}/p! . \tag{5}$$

The signal and interference terms are therefore of the same order if the storage ratio α is finite.

Since the quantity

$$Q = \prod_i \theta(R_i) \tag{6}$$

is equal to one if the vector is perfectly stored and vanishes otherwise, the expectation of Q averaged over all other input patterns, $T=\langle Q\rangle$, gives the perfect storage probability. Clearly, there are correlations between the R_i at different sites i because the interaction matrix $J_{i_1,\ldots,i_p}$ is symmetric. If firstly, however, these correlations are ignored, then

$$T=\langle\theta(R_i)\rangle^N=\exp\left[N\ln\left(\int_{-(p/2\alpha)^{1/2}}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\right)\right] \tag{7}$$

and expansion of this equation for small α/p implies that the exact storage probability is finite provided

$$n\leqslant\frac{N^{p-1}}{2(\ln N)(p-1)!}. \tag{8}$$

It will turn out that equation (8) also holds for the exact calculation of T (for large N) which includes the correlations.

The increase in storage capacity as a function of N as p is increased is, however, compensated by an increase in the computer memory required to store the interaction matrix $J_{i_1,\ldots,i_p}$. A measure of how well the network stores information as a function of p is the ratio of the number of bits needed to store the patterns to the amount of computer memory required to store the matrix and can be defined depending on whether or not the symmetry of the $J_{i_1,\ldots,i_p}$ is used in the storage,

$$2\alpha=\frac{\text{number of patterns}\times\text{number of nodes}}{\text{number of independent } J_{i_1,\ldots,i_p}} \tag{9a}$$

$$a=\frac{2\alpha}{p!}=\frac{\text{number of patterns}\times\text{number of nodes}}{\text{total number of } J_{i_1,\ldots,i_p}} \tag{9b}$$

respectively. In terms of α, the perfect storage capacity increases linearly with p,

$$\alpha<p/4\ln N \tag{10a}$$

while the perfect storage capacity in terms of a decreases with p,

$$a<\frac{1}{2(p-1)!\ln N}. \tag{10b}$$

The exact calculation (for large N) of $T=\langle Q\rangle$ which includes the correlations between different R_i can be done in a similar way to the two-spin calculation (Bruce *et al* 1987, Gardner 1986) by writing an integral representation of the θ function in equation (6),

$$\theta(R_i)=\int_0^{\infty}d\mu_i\int_{-\infty}^{\infty}\frac{d\tau_i}{2\pi}\exp[i(\mu_i-R_i)\tau_i]. \tag{11a}$$

The averages for the other patterns $\{\xi_i^{\mu}\}$ can be done more easily than for $p=2$ since the integrand is a product over these patterns and it is necessary to keep only up to second-order terms in the expansion of the exponential for each pattern μ; higher-order terms are of higher order in $N^{(2-p)/2}$. These higher-order terms correspond to correlations between different $J_{i_1,\ldots,i_p}$ other than those implied by the symmetry of the matrix and these are relevant for large N only if $p=2$. The integrals over the variables $\{\mu_i\}$ and $\{\tau_i\}$ can then be done by introducing an integration variable t which

decouples the integrations of μ_i and τ_i over the sites. T can then be expressed as a saddle point of the integral over t,

$$T=\exp\left(N\left\{\min_t\left[-\tfrac{1}{2}t^2+\ln\left(\int_{\{1-t[(2\alpha/p)(p-1)]^{1/2}\}(2\alpha/p)^{-1/2}}^{\infty}\frac{d\mu}{\sqrt{2\pi}}e^{-\mu^2/2}\right)\right]+O\left(\frac{1}{N}\right)\right\}\right). \tag{11b}$$

Solution of this saddle-point equation gives an expression which is in general different from the approximate expression (7). However, for small α/p one recovers equation (7) and so the storage capacity given by equations (8) and (10) is exact (for large N).

3. Thermodynamics of the large-p limit

In the next two sections, the thermodynamics of the model for finite values of the storage ratio α will be discussed. Physically, the zero-temperature limit gives information about the existence of correlated states which closely represent a particular input pattern while a finite temperature simulates the effect of noise on these results.

Firstly, some properties of the energies of spin configuration will be considered. In the large-p limit this will be used to calculate the density of states as a function of energy. The partition function of the model can then be obtained exactly in a similar way to the solution of the random energy (Derrida 1980, 1981) and a generalisation of it (Mottishaw 1986). The limit $p\to\infty$ is taken after the limit $N\to\infty$.

Provided the patterns have only correlations with one another of order $1/\sqrt{N}$, then any spin configuration $C=\{S_i\}$ has a macroscopic overlap (i.e. an overlap which is finite as $N\to\infty$),

$$m_\mu=\frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu S_i \tag{12}$$

with a finite subset Γ of patterns μ and a microscopic overlap (of order $1/\sqrt{N}$) with all patterns in the complement of this set, $\bar{\Gamma}$, where the number of patterns in $\bar{\Gamma}$ is of order N^{p-1} for large N.

From equation (1), the energy per site of configuration C is

$$\varepsilon_c(\{l_\mu\})=-\sum_{\mu\in\Gamma}l_\mu-\frac{p!}{N^p}\sum_{\mu\in\bar{\Gamma}}\sum_{i_1<\ldots<i_p}\xi_{i_1}^\mu\ldots\xi_{i_p}^\mu S_{i_1}\ldots S_{i_p} \tag{13}$$

where $l_\mu=m_\mu^p$ for $\mu\in\Gamma$ for a given realisation of the patterns.

Consider the set of patterns $\mu\in\Gamma$ to be fixed and consider the subset of configurations with fixed Γ and fixed values of l_μ for $\mu\in\Gamma$. Then the energies ε of these configurations vary as a function of the other patterns and the probability distribution of ε,

$$P(\varepsilon,\{l_\mu\})=\langle\delta(\varepsilon-\varepsilon_c(\{l_\mu\}))\rangle$$

(where $\langle\ \rangle$ means an average over patterns $\mu\in\bar{\Gamma}$) is

$$P(\varepsilon,\{l_\mu\})=\left(\frac{N}{4\pi\alpha}\right)^{1/2}\exp\left[-\frac{N}{4\alpha}\left(\varepsilon+\sum_{\mu\in\Gamma}l_\mu\right)^2\right] \tag{14}$$

for large N.

Expression (14) is obtained by writing P in terms of its Fourier transform which allows the average of the product of exponentials over independent patterns to be factorised,

$$P(\varepsilon,\{l_\mu\})=\int\frac{\mathrm{d}k}{(2\pi/N)^{1/2}}\exp\left[\mathrm{i}kN\left(\varepsilon+\sum_{\mu\in\Gamma}l_\mu\right)\right]$$
$$\times\left\langle\exp\left(\mathrm{i}k\frac{p!}{N^{p-1}}\sum_{i_1<\ldots<i_p}\xi_{i_1}\ldots\xi_{i_p}S_{i_1}\ldots S_{i_p}\right)\right\rangle^{2\alpha N^{p-1}/p!}.$$

The average can be written for large N in terms of a function $f(k(p!N^{2-p})^{1/2})$,

$$\left\langle\exp\left(\mathrm{i}k\frac{p!}{N^{p-1}}\sum_{i_1<\ldots<i_p}\xi_{i_1}\ldots\xi_{i_p}S_{i_1}\ldots S_{i_p}\right)\right\rangle=\exp[f(k(p!N^{2-p})^{1/2})]$$

where

$$f(x)=1-\tfrac{1}{2}x^2+\mathrm{O}(x^3)$$

can be obtained by expanding the exponential and averaging each term. Then

$$P(\varepsilon,\{l_\mu\})=\int\frac{\mathrm{d}k}{(2\pi/N)^{1/2}}$$
$$\times\exp\left[N\left(\mathrm{i}k\left(\varepsilon+\sum_{\mu\in\Gamma}l_\mu\right)-k^2\alpha+\mathrm{O}((kN^{(2-p)/2})^3)N^{p-2}\alpha\right)\right].$$

The saddle point over k at $k_0=(\mathrm{i}/\alpha)(\varepsilon+\Sigma_{\mu\in\Gamma}l_\mu)$ then yields (14) for large N provided $p>2$ with corrections of order $N^{(2-p)/2}$ in N.

One can also calculate the joint probability distribution for two configurations, C_1 and C_2, with macroscopic overlaps $\{l_\mu^1\}$, $\mu\in\Gamma_1$, $\{l_\mu^2\}$, $\mu\in\Gamma_2$, as a function of all patterns μ in the set $\bar{\Gamma}_{1,2}=\bar{\Gamma}_1\cap\bar{\Gamma}_2$. The distribution is (for large N and ε_1 and $\varepsilon_2\sim1$)

$$P_{12}(\varepsilon_1,\{l_\mu^1\},\varepsilon_2,\{l_\mu^2\})$$
$$=\frac{1}{4\pi\alpha N(1-q^p)^{1/2}}\exp\left[-\frac{N}{4\alpha(1+q^p)}\left(\varepsilon_1+\varepsilon_2+\sum_{\mu\in\Gamma_1}l_\mu^1+\sum_{\mu\in\Gamma_2}l_\mu^2\right)^2\right.$$
$$\left.-\frac{N}{4\alpha(1-q^p)}\left(\varepsilon_1-\varepsilon_2+\sum_{\mu\in\Gamma_1}l_\mu^1-\sum_{\mu\in\Gamma_2}l_\mu^2\right)^2\right] \tag{15}$$

where $q=1/N\,\Sigma_i S_i^{(1)}S_i^{(2)}$ is the overlap between configurations 1 and 2.

In the limit $p\to\infty$, the energy levels of the model become independent random variables with mean $-\Sigma_{\mu\in\Gamma}l_\mu$ since equation (15) implies that $P_{12}(\varepsilon_1,\varepsilon_2)$ tends to $P(\varepsilon_1)P(\varepsilon_2)$ for any pair of configurations C_1 and C_2 which are not fully correlated with one another and, more generally, one can show that for any set of distinct configurations $C_1,\ldots,C_m$ and any integer m, the joint probability distribution of their energies factorises:

$$P_{1,2,\ldots,m}(\varepsilon_1,\ldots,\varepsilon_m)=\prod_{i=1}^{m}P(\varepsilon_i). \tag{16}$$

As $p\to\infty$, it is necessary to consider only states which are either fully correlated with a specific input pattern μ_0 ($m_{\mu_0}=1$, $m_\mu=0$ for $\mu\neq\mu_0$) or states which are uncorrelated with any pattern ($m_\mu=0$ for all μ) since $l_\mu=m_\mu^p=1$ if $m_\mu=1$ and $l_\mu=0$ otherwise and since the distributions $P(\varepsilon_1,\ldots,\varepsilon_m)$ depend only on $\{l_\mu^i\}$, $i=1,\ldots,m$.

States which are specific mixtures of input patterns which appear in the two-spin calculations (Amit *et al* 1985a) are therefore degenerate with the uncorrelated states for $p \to \infty$.

The partition function

$$Z = \operatorname{Tr}_{\{S_i\}} e^{-\beta \mathcal{H}} \tag{17}$$

where β is the inverse temperature, can thus be written for large N as

$$Z = n\, e^{-N\beta} + \int_{-\infty}^{\infty} d\varepsilon\, \Omega(\varepsilon)\, e^{-\beta\varepsilon N} \tag{18}$$

where the first term comes from the input patterns which all have energy -1 and the second term comes from the uncorrelated states, and $\Omega(\varepsilon)\, d\varepsilon$ is the number of configurations with energies between ε and $\varepsilon + d\varepsilon$.

In order to determine the phase diagram, it is necessary to know the density of states $\Omega(\varepsilon)$ for a typical sample for large N. The expectation of $\Omega(\varepsilon)\, d\varepsilon$ over all patterns is given for large N by (from equation (14))

$$\langle \Omega(\varepsilon)\, d\varepsilon \rangle = \left(\frac{N}{4\pi\alpha}\right)^{1/2} 2^N \exp(-N\varepsilon^2/4\alpha)\, d\varepsilon. \tag{19}$$

If $d\varepsilon$ is taken to be of order $1/N^{\omega}$, where $0 < \omega < 1$, then expression (19) is exponentially large provided

$$|\varepsilon| < (4\alpha \ln 2)^{1/2}.$$

Therefore, since the energy levels are independent random variables, the typical value of the number of states is equal to its expectation, $\Omega(\varepsilon)\, d\varepsilon = \langle \Omega(\varepsilon)\, d\varepsilon \rangle$, with Gaussian fluctuations $\sim \sqrt{\Omega(\varepsilon)\, d\varepsilon}$. However, if $|\varepsilon| > (4\alpha \ln 2)^{1/2}$ then $\langle \Omega(\varepsilon) \rangle\, d\varepsilon$ is exponentially small in N. Since $\Omega(\varepsilon)\, d\varepsilon$ is an integer, this means that, for almost all samples, $\Omega(\varepsilon)\, d\varepsilon = 0$ and that $\Omega(\varepsilon)\, d\varepsilon = 1$ with a probability which is exponentially small in N. Thus $\Omega(\varepsilon)\, d\varepsilon = 0$ for a typical sample if $|\varepsilon| > (4\alpha \ln 2)^{1/2}$ and so

$$\varepsilon_0 = -(4\alpha \ln 2)^{1/2} \tag{20}$$

is the ground-state energy for the uncorrelated states. Thus

$$Z = n\, e^{-N\beta} + \int_{-(4\alpha \ln 2)^{1/2}}^{(4\alpha \ln 2)^{1/2}} \left(\frac{N}{4\pi\alpha}\right)^{1/2} 2^N \exp(-N\varepsilon^2/4\alpha) \exp(-\beta\varepsilon N)\, d\varepsilon \tag{21}$$

for large N.

The system has three phases depending on which contribution to Z dominates equation (21). If $T > T_g = (\alpha/\ln 2)^{1/2}$, then for large N the integral in equation (21) is dominated by its saddle point; this is a paramagnetic phase with free energy per site,

$$f = -(T \ln 2 + \alpha/T). \tag{22}$$

At $T = T_g$, there is a spin-glass transition for the uncorrelated states where the system simply falls into a set of states which have a finite energy difference from the ground state, and so

$$f = -(4\alpha \ln 2)^{1/2}. \tag{23}$$

In this phase, the integral is dominated by its lower endpoint. The ferromagnetic phase where the input patterns dominate has free energy

$$f = -1 \tag{24}$$

and the phase boundary between this phase and the paramagnetic phase is given by

$$T_p = \frac{1}{2\ln 2} + \left(\frac{1}{4(\ln 2)^2} - \frac{\alpha}{\ln 2} \right)^{1/2} \tag{25}$$

while the boundary between the spin-glass and ferromagnetic phases determines

$$\alpha = 1/4 \ln 2. \tag{26}$$

which is independent of temperature. The phase diagram is shown in figure 1.

Since the ferromagnetic phase exists throughout the phase diagram,

$$\alpha_c \to \infty \qquad \text{as} \qquad p \to \infty \tag{27}$$

for all temperatures and throughout this phase it is completely correlated with the input pattern. It will turn out that the ferromagnetic state is metastable in a region above the spin-glass transition temperature line T_g for all $p > 2$ and, since $\alpha_c(T)$ increases with p, this means that the stability of the input pattern with respect to noise increases with p and that the energy barriers tend to infinity as p tends to infinity. Another difference from $p = 2$ is the existence of a ferromagnetic/paramagnetic transition line.

In order to understand the replica symmetry breaking for the $p \to \infty$ limit and to formulate the model for finite values of p, the replica method will be used.

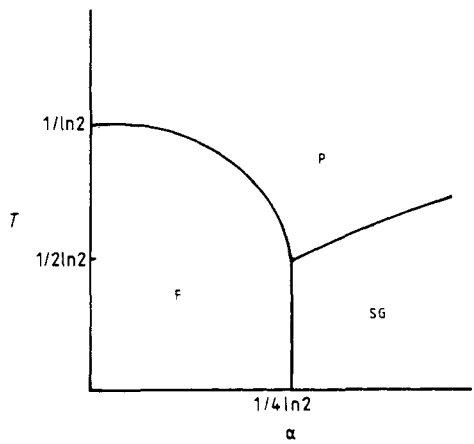

Figure 1. The phase diagram for $p \to \infty$.

4. Replica formalism of the thermodynamics

The free energy is obtained from the expression

$$F = -\lim_{l \to 0} \frac{\langle Z^l \rangle - 1}{l} \tag{28}$$

where

$$\langle Z^l \rangle = \left\langle \operatorname{Tr}_{\{S_i^\gamma\}} \exp\left(-\beta \sum_{\gamma=1}^{l} \mathscr{H}_\gamma \right) \right\rangle \tag{29}$$

and $\mathcal{H}_\gamma$ is the Hamiltonian of equation (1) defined for the spin configuration $\{S_i^\gamma\}$ and $\bar{\Gamma}$ is the set of patterns (with $O(N^{p-1})$ elements) which do not belong to Γ_γ for any replica γ. For large N, one can neglect microscopic overlap terms in $\mathcal{H}_\gamma$ (i.e. $\mu \notin \Gamma_\gamma$) which come from patterns which belong to Γ_δ for some $\delta \neq \gamma$.

Then,

$$\langle Z^l \rangle = \Bigg\langle \sum_{\substack{\text{all possible choices} \\ \text{of } \{\Gamma_\gamma,\ \gamma=1,\dots,l\}}} \operatorname{Tr}_{\{S_i^\gamma\}} \Bigg\{ \prod_{\gamma=1}^{l} \prod_{\mu\in\Gamma_\gamma} \int \frac{\mathrm{d}m_\mu^\gamma}{(2\pi/\beta N)} \int \mathrm{d}k_\mu^\gamma \times \prod_{\gamma<\delta} \int \frac{\mathrm{d}q^{\gamma\delta}}{(2\pi/\beta^2 N)} \int \mathrm{d}r^{\gamma\delta} \exp\Bigg[\beta \sum_{\gamma=1}^{l} \sum_{\mu\in\Gamma_\gamma} (N m_\mu^\gamma - \sum \xi_i^\mu S_i^\gamma) k_\mu^\gamma - \beta^2 \alpha \sum_{\gamma<\delta} \Big(N q^{\gamma\delta} - \sum_i S_i^\gamma S_i^\delta \Big) r^{\gamma\delta} \Bigg] \times \exp\Bigg[\beta \Bigg(\sum_{\mu\in\Gamma_\gamma} (m_\mu^\gamma)^p + \sum_{\gamma=1}^{l} \frac{p!}{N^{p-1}} \sum_{i_1<\dots<i_p} \sum_{\mu\in\Gamma} \xi_{i_1}^\mu \dots \xi_{i_p}^\mu S_{i_1}^\gamma \dots S_{i_p}^\gamma \Bigg) \Bigg] \Bigg\} \Bigg\rangle. \tag{30}$$

Each term in the sum over possible choices of Γ_γ then corresponds to a different way in which the system singles out as $N\to\infty$ a specific Γ_γ for each replica γ. Since the last term in equation (30) is a product over patterns $\mu \in \bar{\Gamma}$, the exponential may be expanded for each pattern and the average over the patterns performed. For a given pattern μ,

$$\begin{aligned} &\Bigg\langle \exp\Bigg(\beta \sum_{\gamma=1}^{l} \frac{p!}{N^{p-1}} \sum_{i_1<\dots<i_p} \xi_{i_1}^\mu \dots \xi_{i_p}^\mu S_{i_1} \dots S_{i_p} \Bigg) \Bigg\rangle \\ &\quad = 1 + \beta \sum_{\gamma=1}^{l} \frac{p!}{N^{p-1}} \sum_{i_1<\dots<i_p} \xi_{i_1}^\mu \dots \xi_{i_p}^\mu S_{i_1} \dots S_{i_p} + \tfrac{1}{2}\beta^2 \Bigg(\frac{p!}{N^{p-1}} \Bigg)^2 \\ &\qquad \times \Bigg(\sum_{\gamma=1}^{l} \sum_{i_1<\dots<i_p} \xi_{i_1}^\mu \dots \xi_{i_p}^\mu S_{i_1} \dots S_{i_p} \Bigg)^2 + \cdots \\ &\quad = 1 + \frac{\beta^2}{2} \frac{p!}{N^{p-1}} \frac{1}{p!} \sum_{\gamma,\delta} \Bigg(\sum_i S_i^\gamma S_i^\delta \Bigg)^p + O(N^{(3-p)/2}) \end{aligned} \tag{31}$$

since the linear term averages to zero and the number of terms contributing to the second average is $N^p/p!$ for large N. The number of terms contributing to the kth average is of order $(N^{p/2})^k$ in N and so that the kth term in the expansion is of order $N^{k(1-p/2)}$. Since there are order N^{p-1} patterns, the higher-order terms in the expansion may be neglected provided $p>2$ (if $p=2$ the series may be resummed to give the expression of Amit *et al* (1985b)). These higher-order terms correspond to correlations in the $J_{i_1,\dots,i_p}$ other than those implied by the symmetry of the matrix.

Then the free energy is obtained from the saddle point of the integrals over m_μ^γ, k_μ^γ, $q^{\gamma\delta}$, $r^{\gamma\delta}$:

$$\begin{aligned} \frac{\beta F}{N} = -\tfrac{1}{2}\beta^2\alpha - \lim_{l\to 0} \operatorname{Ext} \frac{1}{l} \Bigg[& 2\beta^2\alpha \sum_{\gamma<\delta} (q^{\gamma\delta})^p - \alpha\beta^2 \sum_{\gamma<\delta} q^{\gamma\delta} r^{\gamma\delta} \\ & + \beta \sum_{\mu\in\Gamma_\gamma} (m_\mu^\gamma)^p - \beta \sum_{m\in\Gamma_\gamma} k_\mu^\gamma m_\mu^\gamma \\ & + \ln \Bigg\langle \operatorname{Tr} \exp\Bigg(\alpha\beta^2 \sum_{\gamma<\delta} S^\gamma S^\delta r^{\gamma\delta} + \beta \sum_{\mu\in\Gamma_\gamma} \xi^\mu k_\mu^\gamma S^\gamma \Bigg) \Bigg\rangle \Bigg] \end{aligned} \tag{32}$$

when the extremum means a minimisation of F with respect to m_μ^γ and $r^{\gamma\delta}$ and a maximisation with respect to k_μ^γ and $q^{\gamma\delta}$.

It will be assumed that there is a macroscopic overlap with only one pattern (pattern 1). Higher-energy mixture states can be obtained by assuming that there are macroscopic overlaps with more than one pattern. Secondly, the replica symmetric ansatz will be made,

$$q^{\gamma\delta} = q \qquad r^{\gamma\delta} = r \qquad k_1^\gamma = k \qquad m_1^\gamma = m \tag{33}$$

which leads to the following mean-field equations:

$$m = \int_{-\infty}^{\infty} Dz \tanh \beta(k + \sqrt{\alpha r}\, z)$$

$$q = \int_{-\infty}^{\infty} Dz \tanh^2 \beta(k + \sqrt{\alpha r}\, z)$$

$$k = pm^{p-1}$$

$$r = 2pq^{p-1}$$

where

$$Dz = \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2}. \tag{34}$$

At all temperatures, these equations have a paramagnetic solution,

$$r = q = m = k = 0 \tag{35}$$

with free energy

$$\beta F/N = -\ln 2 - \beta^2 \alpha \tag{36}$$

and this solution will be the correct one at high temperatures.

There are also other solutions to equations (34). In particular, at zero temperature one has

$$m = 2 \operatorname{erf}[(p/2\alpha)^{1/2} m^{p-1}] \tag{37}$$

and $q = 1$, $r = 2p$, $k = pm^{p-1}$ provided $k \neq 0$. Equations (37) have a solution with non-zero m and k provided $\alpha < \alpha_c$ where the largest value of m is stable with respect to variations in the non-spin-glass order parameters m and k for any value of α, whereas the other solution is always unstable. There is a first-order transition at α_c where the two fixed points merge and annihilate. The order parameter m jumps from a finite value to zero at α_c. The two-spin model also has a first-order transition in α at zero temperature (Amit *et al* 1985b). However, the mechanism is different in this case. For $p > 2$, the transition is generated by second-order correlations between different $J_{i_1,\ldots,i_p}$; higher-order correlations vanish as $N \to \infty$ for the reasons explained after equation (31). However, for $p = 2$, these higher-order terms remain finite for large N and must be included. Equation (37) is therefore incorrect for $p = 2$ because it includes only terms coming from correlations due to the symmetry of the matrix J_{ij}. Although the equation has a non-trivial solution for $\alpha < \alpha_0 = 2/\pi \sim 0.63$ for $p = 2$, the transition is second order; as $\alpha \to 2/\pi$, the value of m tends to zero. This means that the higher-order correlations between different J_{ij} are needed to generate the first-order transition at $\alpha = \alpha_c \sim 0.14$ and to remove the second-order transition at α_0 for $p = 2$.

As $p \to \infty$, the solution is $m = 1$, in agreement with the exact solution, and is metastable throughout the T, α plane of the phase diagram. The replica symmetric ansatz also predicts the way in which $\alpha_c \to \infty$ as $p \to \infty$,

$$\alpha_c \sim p/4 \ln p \tag{38}$$

at zero temperature (from equation (34)). For $p = 3$ and 4, table 1 gives the replica symmetric predictions for α_c and the values of m at this value of α. The fraction of errors $\frac{1}{2}(1 - m_c)$ at α_c is quite small for both $p = 3$ and 4 (although it is larger than for $p = 2$). This means that the quality of recall is good below α_c. Replica symmetry breaking effects will increase α_c at low temperatures and since m_c is smaller these effects will probably be larger than for $p = 2$.

The equations for states which have macroscopic overlaps with more than one pattern can also be written. At zero temperature, these states are the same as those found by Amit *et al* for the two-spin model. However, since the signal term depends on $\Sigma_{\mu \in \Gamma} m_\mu^p$ their critical temperature and storage ratios decrease with p and their effect vanishes completely in the large-p limit.

In order to obtain values for α_1, it is necessary to consider spin-glass solutions of equation (32). If k_μ^γ is set equal to zero in equation (30), then the free energy is the same as that for the p-spin glass with Gaussian interactions (Gross and Mezard 1984, Gardner 1985) where the temperature is simply rescaled by a factor $\sqrt{\alpha}$ and so spin-glass and paramagnetic solutions of (32) are the same as those found for this model. Solutions of the Sherrington-Kirkpatrick model (1975, 1978) and spin-glass solutions of the Hopfield model ($p = 2$) are not related in this simple way because of the higher-order terms in equation (31). Spin-glass transitions in p-spin models for $p > 2$ are qualitatively different from the $p = 2$ model. Firstly, the paramagnetic solution (35) and (36) is stable at all temperatures with respect to fluctuations in both spin-glass and ferromagnetic order parameters. However, it is not the correct solution at low temperatures; the entropy derived from equation (36) becomes negative below a temperature

$$T_E = (\alpha / \ln 2)^{1/2}$$

and so there must be a phase transition at a temperature $T_g > T_E$ to a new low-temperature phase. Secondly, low-temperature replica symmetric solutions do not exist at sufficiently high temperatures. However, if one assumes one replica symmetry breaking, a solution does exist for all values of p.

For $p \to \infty$, this solution is exact at all temperatures below T_E (Gross and Mezard 1984); the Parisi order parameter function $q(x)$ is a step function with lower value 0 and upper value 1 and the break point x_0 behaves as T/T_E in the low-temperature phase and the phase transition is at $T = T_E$ in agreement with the exact solution. Physically, the low-temperature phase consists of a set of valleys whose energies have

Table 1. Values of α_c, a_c, m_c and α_1 as a function of p.

p	α_c	a_c	m_c	α_1
2	0.14	0.14	0.97	0.05
3	0.38	0.13	0.855	0.38
4	0.395	0.10	0.94	0.38
$p \to \infty$	$p/4 \ln p$	$[2 \ln p(p-1)!]^{-1}$	1	0.36

a finite difference from the ground-state energy. Since the valleys have zero size in the limit $p \to \infty$ the upper value of $q(x)$ is 1 and, since the energy levels of the model are independent random variables, the overlap between the valleys is always zero.

For p finite (Gardner 1985) a one-replica-breaking solution (with $m = k = 0$) exists, the phase transition temperature T_g is larger than T_E and the order parameter function is a step function with lower value 0 and upper value $q_1(T, p) < 1$ since the valleys now have finite size. The transition is similar to the $p \to \infty$ transition and the new solution is stable throughout a whole phase but (for finite p) becomes unstable at a lower transition temperature $T_2(p)$. This solution may therefore be used to obtain an approximate value for the energy transition α_1 at $T = 0$ by comparison of its energy with that of the replica symmetric ferromagnetic solution, and the values for $p = 3$ and 4 are given in table 1. For $p = 3$, $\alpha_1 = \alpha_c$ since the ferromagnetic solution is always of lower energy while, for $p = 4$, α_1 is slightly smaller than α_c in this approximation.

The phase diagram is qualitatively similar to the $p \to \infty$ limit (although $\alpha_c(T)$ is now finite and the ferromagnetic solution is not completely correlated with the input pattern); the ferromagnetic solution remains metastable above the spin-glass transition temperature T_g and $\alpha_c(T)$ increases with p which means that the stability of the patterns also increases with p; the ferromagnetic/spin-glass transition does not approach $\alpha = 0$ as in the case $p = 2$.

5. Conclusions

In summary, higher-spin correlations increase the number of patterns which can be stored. However, this is compensated by an increase in computer memory by the same power of N. This means that it is useful to define the ratios α and a (equations 9(a), (b)) of the number of bits stored in the patterns to computer memory, depending on whether or not the symmetry of the matrix is used in the storage. In terms of both mean-field calculations for exact storage capacity and for the thermodynamics, the case $p = 2$ is special, correlations between $J_{i_1,\ldots,i_p}$ other than those implied by the symmetry of the matrix being relevant in this case. The structure of the finite-temperature phase diagram is also different. For all values of p, the overlap with the input pattern m remains quite close to 1 (at zero temperature) even at α_c and so the fraction of errors $\frac{1}{2}(1 - m)$ is always small and thus the quality of recall is good. For $p > 3$, one has qualitatively the following results: the maximum storage fraction α_c (equation 9(a)) for which a correlated metastable state exists increases with p if the symmetry of the $J_{i_1,\ldots,i_p}$ is used in the storage (and in the replica symmetric approximation $\alpha_c \sim p/4 \ln p$ as $p \to \infty$) whereas the maximum of the ratio a where the symmetry is not used (equation 9(b)), a_c, decreases with p ($a_c \sim 1/2(p-1)! \ln p$). The increase with p in the maximum temperature at which a metastable correlated state exists implies an increase in the heights of energy barriers and therefore an increase in the stability of the patterns with respect to noise.

A disadvantage of increasing the value of p is that associative memory (the size of the basin of attraction of the input patterns) decreases (for $p > 3$). Although α_c increases with p, the ability of the system to recognise patterns with a finite fraction of errors decreases. This can be seen firstly because the number of spurious metastable states increases with p and tends to $2^{N(1+O(1/N))}$ as $p \to \infty$—nearly all states are metastable in this limit. Secondly, from equation (15) correlations between the energy levels are determined by the value of q^p and so the size of a valley should behave approximately

as $1/p$ for large p. The effect of specific mixture states, however, decreases with p and these states disappear completely at $p \to \infty$.

From table 1, the three-spin model has a value of α_c approximately three times larger than that of the two-spin model and so both models have approximately the same values of a_c, although the fraction of errors at a_c is greater in the three-spin model (at least in the replica symmetric approximation). The approximate equality of the values of a_c agrees with numerical results (Maxwell *et al* 1986a, b, Psaltis and Park 1986). Replica symmetry breaking effects for the ferromagnetic solution should increase α_c and a_c and be larger than for the two-spin model, although, since m is quite close to 1, this effect could still be small. Further replica breaking for both spin-glass and ferromagnetic solutions will also change finite p predictions for α_1 (although the value is exact as $p \to \infty$).

Another result is that, at least in the case of the rather simple large-p limit (where the domains of attraction of input patterns vanish and the ferromagnetic states are completely correlated with the patterns), replica symmetry breaking introduces no new qualitative effects with respect to zero-temperature phase transitions.

It would be interesting to extend these results to correlated patterns. In particular, multiconnected interactions should provide a way of reducing the effect of unwanted specific mixture states which appear in the two-spin calculations (Amit *et al* 1987).

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